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Coclass theory for nilpotent semigroups via their associated algebras

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ABSTRACT

Coclass theory has been a highly successful approach towards the investigation and classification of finite nilpotent groups. Here we suggest a variation of this approach for finite nilpotent semigroups: we propose to study coclass graphs for the contracted semigroup algebras of nilpotent semigroups. We exhibit a series of conjectures on the shape of these coclass graphs. If these are proven, then this reduces the classification of nilpotent semigroups of a fixed coclass to a finite calculation. We show that our conjectures are supported by the nilpotent semigroups of coclass 0 and 1. Computational experiments suggest that the conjectures also hold for the nilpotent semigroups of coclass 2 and 3.

1. Introduction

A semigroup O or an associative algebra O is *nilpotent* if there exists an integer c so that every product of $c+1$ elements equals zero. The least integer c with this property is the *class* $\mathrm{cl}(O)$ of O; equivalently, the class of O is the length of series of powers

$$O > O^2 > \cdots > O^c > O^{c+1} = \{0\}.$$

The *coclass* of a finite nilpotent semigroup O with n non-zero elements or a finite dimensional nilpotent algebra O of dimension n is defined as $\mathrm{cc}(O) = n - \mathrm{cl}(O)$.

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For a semigroup S and a field K we denote by $K[S]$ the semigroup algebra defined by K and S. This is an associative algebra of dimension $|S|$. If S has a zero element z, then the subspace U of $K[S]$ generated by z is an ideal in $K[S]$. We call $K[S]/U$ the *contracted semigroup algebra* defined by K and S and denote it by KS. If S is a finite nilpotent semigroup, then KS is a nilpotent algebra of the same class and coclass as S.

Our first aim in this note is to suggest a general approach towards a classification up to isomorphism of nilpotent semigroups of a fixed coclass. For this purpose we choose an arbitrary field K and we define a directed labelled graph $\mathcal{G}_{r,K}$ as follows: the vertices of $\mathcal{G}_{r,K}$ correspond one-to-one to the isomorphism types of algebras KS for the nilpotent semigroups S of coclass r; two vertices A and B are adjoined by a directed edge $A \to B$ if $B/B^c \cong A$, where c is the class of B; each vertex A in $\mathcal{G}_{r,K}$ is labelled by the number of isomorphism types of semigroups S of coclass r with $A \cong KS$. Illustrations of parts of such graphs appear as Fig. 1 on page 498 and as Fig. 2 on page 500.

We have investigated various of the graphs $\mathcal{G}_{r,K}$ and we observed that all these graphs share the same general features. We formulate a sequence of conjectures and theorems describing these features. If our conjectures are proved, then this would provide a new approach towards the classification and investigation of nilpotent semigroups by coclass. In particular, it would show how the classification of the infinitely many nilpotent semigroups of a fixed coclass reduces to a finite calculation.

As a second aim, we exhibit some graphs $\mathcal{G}_{r,K}$ explicitly to illustrate our conjectures. We have determined the graphs $\mathcal{G}_{0,K}$ and $\mathcal{G}_{1,K}$ for all fields K using the classification of the nilpotent semigroups of small coclass in [2,3]; see Sections 5 and 6. Further, we investigated the graphs $\mathcal{G}_{2,K}$ and $\mathcal{G}_{3,K}$ for some finite fields K using computational methods based on [6] to solve the isomorphism problem for nilpotent associative algebras over finite fields; see Section 7.

One can also define a directed graph $\mathcal{G}_r$ whose vertices correspond one-to-one to the isomorphism types of semigroups of coclass r. While the graphs $\mathcal{G}_r$ are also of interest, they do not exhibit the same general features as $\mathcal{G}_{r,K}$. We compare $\mathcal{G}_r$ and $\mathcal{G}_{r,K}$ briefly in Section 9.

The idea of using coclass for the classification of nilpotent algebraic objects was first introduced by Leedham-Green and Newman [10] for nilpotent groups. We refer to the book by Leedham-Green and McKay [9] for background and many details on the results in the group case. Various details of the approach taken here are similar to the concepts in group theory. In particular, the idea of searching for periodic patterns in coclass graphs as used below also arises in group theory; we refer to [5,7] for details. But a nilpotent semigroup is not a group and hence the coclass theories for groups and semigroups are independent.

2. Coclass conjectures for semigroups

In this section we investigate general features of the graph $\mathcal{G}_{r,K}$ for $r \in \mathbb{N}_0$ and arbitrary field K.

By construction, every connected component of $\mathcal{G}_{r,K}$ is a rooted tree. Using basic results on nilpotent semigroups (see [3, Lemma 2.1]) one readily shows that $2r$ is an upper bound for the dimension of a root (that is, the dimension of the corresponding algebra) in $\mathcal{G}_{r,K}$. Thus $\mathcal{G}_{r,K}$ consists of finitely many rooted trees. We call an infinite path in a rooted tree *maximal* if it starts at the root of the tree.

Conjecture I. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Then the graph $\mathcal{G}_{r,K}$ has only finitely many maximal infinite paths. The number of such paths depends on r but not on K.*

For an algebra A in $\mathcal{G}_{r,K}$ we denote by $\mathcal{T}(A)$ the subgraph of $\mathcal{G}_{r,K}$ consisting of all paths that start at A. This is a rooted tree with root A. We say that $\mathcal{T}(A)$ is a *coclass tree* if it contains a unique maximal infinite path. A coclass tree $\mathcal{T}(A)$ is *maximal* if either A is a root in $\mathcal{G}_{r,K}$ or the parent of A lies on more than one maximal infinite paths.

1 Remark. Conjecture I is equivalent to saying that $\mathcal{G}_{r,K}$ consists of finitely many maximal coclass trees and finitely many other vertices.

We consider the maximal coclass trees in $\mathcal{G}_{r,K}$ in more detail. For a labelled tree $\mathcal{T}$ we denote by $\overline{\mathcal{T}}$ the tree without labels.

Conjecture II. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{r,K}$ with maximal infinite path $A_1 \to A_2 \to \cdots$. Then $\mathcal{T}$ is* weakly virtually periodic: *there exist positive integers l and k so that $\overline{\mathcal{T}}(A_l) \cong \overline{\mathcal{T}}(A_{l+k})$.*

The integers l and k with the property of Conjecture II are the *weak defect* and *weak period* of $\overline{\mathcal{T}}$. Note that they are not unique. Every integer larger than l and every multiple of k are weak defects and weak periods as well, respectively.

Consider a maximal coclass tree $\mathcal{T}$ of $\mathcal{G}_{r,K}$ with maximal infinite path $A_1 \to A_2 \to \cdots$. Suppose that for some l and k there exists a graph isomorphism $\mu : \overline{\mathcal{T}}(A_l) \to \overline{\mathcal{T}}(A_{l+k})$. Then μ defines a partition of the vertices of $\mathcal{T}(A_l)$ into finitely many infinite families: for each vertex B contained in $\mathcal{T}(A_l) \setminus \mathcal{T}(A_{l+k})$ define the infinite family $(B, \mu(B), \mu^2(B), \ldots)$. Hence Conjecture II asserts that the unlabelled tree $\overline{\mathcal{T}}$ can be constructed from a finite subgraph, provided that a weak defect and a weak period are known. This implies that $\mathcal{T}$ has finite width. Conjecture I asserts that these features of maximal coclass trees extend to all of $\mathcal{G}_{r,K}$. We next exhibit an extension of Conjecture II incorporating labels.

Conjecture III. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{r,K}$ with maximal infinite path $A_1 \to A_2 \to \cdots$. Then $\mathcal{T}$ is* strongly virtually periodic: *there exist positive integers l and k, a graph isomorphism $\mu : \overline{\mathcal{T}}(A_l) \to \overline{\mathcal{T}}(A_{l+k})$ and for every vertex B in $\mathcal{T}(A_l) \setminus \mathcal{T}(A_{l+k})$ a rational polynomial f_B so that the label of $\mu^i(B)$ equals $f_B(i)$.*

The integers l and k with the property of Conjecture III are the *strong defect* and *strong period* of $\mathcal{T}$. As in the weak case, they are not unique. Further, every strong defect and strong period are also a weak defect and weak period, but the converse does not hold in general; see Section 6 for an example.

Conjectures I and III suggest the following new approach towards a classification up to isomorphism of all nilpotent semigroups of fixed coclass $r \in \mathbb{N}_0$.

(1) Choose an arbitrary field K and classify the maximal infinite paths in $\mathcal{G}_{r,K}$.
(2) For each maximal infinite path, consider its corresponding coclass tree $\mathcal{T}$ and find a strong defect l, a strong period k and an upper bound d to the degree of the polynomials of the associated families.
(3) For each maximal coclass tree $\mathcal{T}$ with strong defect l, strong period k and bound d:
  (a) Determine the unlabelled tree $\overline{\mathcal{T}}$ up to depth $l + (d+1)k$.
  (b) For each vertex B in the determined part of $\overline{\mathcal{T}}$ compute its label.
(4) Determine the finite parts of $\mathcal{G}_{r,K}$ outside the maximal coclass trees.

Step (1) is discussed further in Section 3 below. For step (2) we hope that a constructive proof of the conjectures posed here may also yield values for strong defect, strong period and bounds for the degrees of the arising polynomials.

Steps (3)(a) and (3)(b) may be facilitated by two algorithms. The first determines up to isomorphism all contracted semigroup algebras B of class $c+1$ with $B/B^{c+1} \cong A$ for any given contracted semigroup algebra A of class c. The second algorithm takes a nilpotent associative algebra A of finite dimension and computes up to isomorphism all semigroups S with $KS \cong A$. Both algorithms reduce to a finite computation if the underlying field K is finite. A practical realisation for the first algorithm in the finite field case may be obtained as variation of the method in [6].

Once steps (1)–(4) have been performed, this would allow us to construct the full graph $\mathcal{G}_{r,K}$ using the graph isomorphism of Conjecture III. The polynomials f_B can be interpolated from the given information, as there are $d+1$ values $f_B(i)$ available.

3. The infinite paths in $\mathcal{G}_{r,K}$

In this section we investigate in more detail the infinite paths in $\mathcal{G}_{r,K}$ for arbitrary $r \in \mathbb{N}_0$ and arbitrary field K. We first provide some background for our constructions.

3.1. Coclass for infinite objects

Let O be a finitely generated infinite semigroup or a finitely generated infinite dimensional associative algebra. Then O^i is an ideal in O for every $i \in \mathbb{N}$ and thus we can form the quotient O/O^i (this is the Rees quotient in the semigroup case [8, page 33]). Every quotient O/O^i is finitely generated of class at most $i-1$ and hence is finite (in the semigroup case) or finite dimensional (in the algebra case). Thus O/O^i has finite coclass $\mathrm{cc}(O/O^i)$. We say that O is *residually nilpotent* if $\bigcap_{i\in\mathbb{N}} O^i = 0$ holds. If O is finitely generated and residually nilpotent, then we define its *coclass* $\mathrm{cc}(O)$ as

$$\mathrm{cc}(O) = \lim_{i\to\infty} \mathrm{cc}\bigl(O/O^i\bigr).$$

The coclass of O can be finite or infinite. It is finite if and only if there exists $i \in \mathbb{N}$ so that $|O^j \setminus O^{j+1}| = 1$ (in the semigroup case) or $\dim(O^j/O^{j+1}) = 1$ (in the algebra case) for all $j \geqslant i$. If we say that O has 'finite coclass', then this implies that O is finitely generated and residually nilpotent.

3.2. Inverse limits of algebras and semigroups

Consider a maximal infinite path $A_1 \to A_2 \to \cdots$ in $\mathcal{G}_{r,K}$ and let $\hat{A} = \prod_{i\in\mathbb{N}} A_i$ be the Cartesian product of the algebras on the path. If A_1 has class c, then A_j has class $j+c-1$ and thus $A_{j+1}/A_{j+1}^{j+c} \cong A_j$ for every $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ we choose an epimorphism $\nu_j : A_{j+1} \to A_j$ with kernel A_{j+1}^{j+c}. We define the *inverse limit* of the algebras on the path as

$$A = \bigl\{(a_1, a_2, \ldots) \in \hat{A} \bigm| \nu_j(a_{j+1}) = a_j \text{ for every } j \in \mathbb{N}\bigr\}.$$

The inverse limit A is an infinite dimensional associative K-algebra which satisfies $A/A^{j+c} \cong A_j$ for every $j \in \mathbb{N}$. Thus A/A^2 is finite dimensional and hence A is finitely generated. It is also residually finite and has coclass r. Further, each algebra on the maximal infinite path can be obtained as a quotient of A and thus A fully describes the considered maximal infinite path. We summarise this as follows.

2 Theorem. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. For every maximal infinite path in $\mathcal{G}_{r,K}$ there exists an infinite dimensional associative K-algebra of coclass r which describes the path.*

Isomorphic algebras of the type considered in Theorem 2 describe the same infinite path. Hence an approach to the classification of the maximal infinite paths in $\mathcal{G}_{r,K}$ is the determination up to isomorphism of the infinite dimensional associative K-algebras A of coclass r whose quotients A/A^j are contracted semigroup algebras for every $j \in \mathbb{N}$. Conjecture I equivalently states that there are only finitely many of these objects up to isomorphism. The following theorem describes these algebras in more detail.

3 Theorem. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Each infinite dimensional associative K-algebra of coclass r which describes an infinite path in $\mathcal{G}_{r,K}$ is isomorphic to a contracted semigroup algebra KS for an infinite semigroup S of coclass r.*

Proof. Let A be an infinite dimensional associative K-algebra of coclass r which describes an infinite path in $\mathcal{G}_{r,K}$. Then there exists $i \in \mathbb{N}$ so that A/A^j is a contracted semigroup algebra of coclass r for every $j \geqslant i$. Each of the quotients A/A^j may be the contracted semigroup algebra for several non-isomorphic semigroups. Our aim is to show that for every $j \geqslant i$ there exists a semigroup S_j so that $A/A^j \cong KS_j$ and $S_j \cong S/S^j$ for an infinite semigroup S of coclass r.

We define a graph $\mathcal{L}$ whose vertices correspond one-to-one to the isomorphism types of semigroups whose contracted semigroup algebra is isomorphic to a quotient A/A^j for some $j \geqslant i$. We connect two semigroups in $\mathcal{L}$ by a directed edge $U \to T$ if $T/T^c \cong U$, where c is the class of T. If a semigroup T satisfies $KT \cong A/A^j$ for some $j > i$, then T has class $j-1$ and $U \cong T/T^{j-1}$ satisfies $KU \cong A/A^{j-1}$. Hence each connected component of $\mathcal{L}$ is a tree with a root of class $i-1$ and coclass r. There is at least one infinite connected component $\mathcal{M}$ of $\mathcal{L}$. By König's Infinity Lemma, see [1, Lemma 8.1.2], the tree $\mathcal{M}$ contains an infinite path, say $M_i \to M_{i+1} \to \cdots$. Let S be the inverse limit of the semigroups on this infinite path. Then S is an infinite semigroup with $S/S^j \cong M_j$ and $KM_j \cong A/A^j$ for every $j \geqslant i$. In particular, the semigroup S has finite coclass r.

It remains to show that S satisfies $KS \cong A$. This follows from the construction of S, as the following diagram is commutative, where upwards arrows denote embeddings of semigroups in their contracted semigroup algebras:

$$\begin{array}{ccccccc} A/A^i & \to & A/A^{i+1} & \to & A/A^{i+2} & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ S/S^i & \to & S/S^{i+1} & \to & S/S^{i+2} & \to & \cdots. \end{array}$$

This completes the proof. $\square$

3.3. *Examples*

Consider the polynomial algebra in one indeterminate and let I_K denote its ideal consisting of all polynomials with zero constant term. Then I_K is an explicit construction for the free non-unital associative algebra on one generator over the field K. It is isomorphic to the contracted semigroup algebra KS with $S \cong (\mathbb{N}_0, +)$ and it has coclass 0. Hence it describes an infinite path in $\mathcal{G}_{0,K}$. In Section 5 we observe that it describes the unique maximal infinite path in $\mathcal{G}_{0,K}$.

Examples of infinite dimensional contracted semigroup algebras of higher coclass can be obtained inductively using the following process. Let S be an infinite semigroup of coclass $r-1$. An *annihilator extension* of S is an infinite semigroup T so that T contains a non-zero element t with $tT = Tt = \{0\}$ and $S \cong T/\langle t\rangle$ (again this is the Rees quotient).

4 Lemma. *Let $r \in \mathbb{N}$ and let S be an infinite semigroup of coclass $r-1$. Each annihilator extension T of S is an infinite semigroup of coclass r.*

Proof. Consider the sequence $T \geqslant T^2 \geqslant T^3 \geqslant \cdots$ and define $c \in \mathbb{N}$ via $t \in T^c \setminus T^{c+1}$. Let $\nu : T \to S$ be the Rees morphism, see [8, page 33]. Then $\nu(t) = 0 = \nu(0)$. As $\nu(T^i) = S^i$, we obtain that $T/T^i \cong S/S^i$ for $1 \leqslant i \leqslant c$ and for $i \geqslant c+1$ we obtain that $|T/T^i| = |S/S^i| + 1$. Thus T is finitely generated and residually nilpotent and it satisfies $\mathrm{cc}(T/T^i) = \mathrm{cc}(S/S^i) + 1$ for $i \geqslant c+1$. Thus T is an infinite semigroup of coclass $\mathrm{cc}(T) = \mathrm{cc}(S) + 1$. $\square$

If A is an infinite dimensional contracted semigroup algebra of coclass $r-1$, then $A = KS$ for an infinite semigroup S of coclass $r-1$. Thus every annihilator extension T of S gives rise to an infinite dimensional contracted semigroup algebra of coclass r.

We exhibit an explicit example for this process. Let L_K denote the 1-dimensional nilpotent algebra of class 1. Then L_K is isomorphic to the contracted semigroup algebra KZ_2, where Z_n is the zero semigroup with n elements. For every $r \in \mathbb{N}$ the algebra

$$M_{K,r} = I_K \oplus \bigoplus_{i=1}^{r} L_K \qquad (1)$$

is an infinite dimensional contracted semigroup algebra of coclass r. As underlying semigroup one can choose the zero union of $(\mathbb{N}, +)$ and Z_{r+1}, that is the semigroup on $\mathbb{N} \cup Z_{r+1}$ in which mixed products equal $0 \in Z_{r+1}$. For $r > 0$ this is an annihilator extension of the zero union of $(\mathbb{N}, +)$ and Z_r corresponding to $M_{K,r-1}$.

We close this section by posing the following question.

5 Question. Does every infinite dimensional algebra which describes an infinite path in $\mathcal{G}_{r,K}$ arise as contracted semigroup algebra for a semigroup which is an annihilator extension?

4. The minimal generator number

A nilpotent semigroup S has a unique minimal generating set $S \setminus S^2$. Its cardinality corresponds to the dimension of the quotient $KS/(KS)^2$ and thus to the minimal generator number of the algebra KS. Hence $KS \cong KT$ implies that the nilpotent semigroups S and T have the same minimal generator number. Further, if two algebras in $\mathcal{G}_{r,K}$ are connected, then they have the same minimal generator number. This allows us to define the subgraph $\mathcal{G}_{r,K,d}$ of $\mathcal{G}_{r,K}$ corresponding to the nilpotent semigroups of coclass r with minimal generator number d.

A nilpotent semigroup of coclass r has at most $r+1$ generators. Thus $\mathcal{G}_{r,K,d}$ is empty for $d \geqslant r+2$ (and also for $d = 1$ if $r > 0$). The extremal case $\mathcal{G}_{r,K,r+1}$ can be described in more detail as the following theorem shows. Recall that Z_n is the zero semigroup with n elements and $M_{K,r}$ is defined in (1).

6 Theorem. *Let $r \in \mathbb{N}_0$ and K an arbitrary field. Then $\mathcal{G}_{r,K,r+1}$ consists of a unique maximal coclass tree with corresponding infinite dimensional algebra $M_{K,r}$. The root of the maximal coclass tree is KZ_{r+2} if $r > 0$ and KZ_1 if $r = 0$.*

Proof. The semigroup Z_{r+2} has $r+2$ elements, minimal generator number $r+1$, class 1 and thus coclass r. If $r > 0$, then Z_{r+2} is the unique semigroup of coclass r and order at most $r+2$ and hence KZ_{r+2} is a root in $\mathcal{G}_{r,K,r+1}$. If $r = 0$, then Z_1 is a root of $\mathcal{G}_{0,K,1}$.

In the following we assume that $r > 0$. The case $r = 0$ is similar and we leave it to the reader. Let S be an arbitrary semigroup of class c such that KS is in $\mathcal{G}_{r,K,r+1}$. We show by induction on $|S|$ that there exists a path from KZ_{r+2} to KS. As Z_{r+2} is the only semigroup of coclass r, order at most $r+2$ and minimal generator number $r+1$, we may assume that $|S| > r+2$. Since $|S \setminus S^2| = r+1$, it follows that $|S^2| = c$ and hence $|S^c| = 2$. Thus S/S^c is a semigroup of coclass r with minimal generator number $r+1$ and with $|S|-1$ elements. Hence there is an edge from $KS/(KS)^c \cong K(S/S^c)$ to KS. By induction, there exists a path from KZ_{r+2} to $K(S/S^c)$ and hence to KS. This proves that $\mathcal{G}_{r,K,r+1}$ is connected.

The infinite dimensional algebra $M_{K,r}$ has coclass r and minimal generator number $r+1$ and it is a contracted semigroup algebra. It defines a maximal infinite path in $\mathcal{G}_{r,K,r+1}$. It remains to show that this maximal infinite path is unique. Let A be an arbitrary infinite dimensional associative algebra of coclass r with $r+1$ generators. Then $\dim(A/A^2) = r+1$ and $\dim(A^i/A^{i+1}) = 1$ for every $i \geqslant 2$. Let $v, w, x \in A$ such that $vwxA^4$ generates A^3/A^4. Then both vwA^3 and wxA^3 generate A^2/A^3 and hence $vw = kwx$ for some $k \in K$. This yields $vwx = kwxx$ and hence x^2A^3 is a generator of A^2/A^3. By induction, it follows that x^iA^{i+1} is a generator of A^i/A^{i+1} for every $i \geqslant 2$. Now choose elements $x_1, \ldots, x_r \in A$ that together with x correspond to a basis of A/A^2. Then these elements generate A. A basis of A^2 has the form $\{x^j \mid j \geqslant 2\}$. Thus for $i \in \{1, \ldots, r\}$,

$$xx_i = \sum_{j=2}^{\infty} k_{ij}x^j \in A^2.$$

We replace x_i by

$$y_i = x_i - \sum_{j=2}^{\infty} k_{ij} x^{j-1}$$

and thus obtain a new minimal generating set $x, y_1, \ldots, y_r$ of A which satisfies $xy_i = 0$ by construction. For $i, j \in \{1, \ldots, r\}$ consider the product $y_i y_j$. Then $y_i y_j = \sum_{h=2}^{\infty} k_h x^h \in A^2$. As $xy_i = 0$, it follows that $xy_i y_j = 0$ and thus $\sum_{h=2}^{\infty} k_h x^{h+1} = 0$. This implies that all coefficients k_h equal 0 and hence $y_i y_j = 0$ for every $i, j \in \{1, \ldots, r\}$. Thus $A \cong M_{K,r}$. □

5. The graph $\mathcal{G}_{0,K}$

The semigroups of coclass 0 are well known; for every order $n \in \mathbb{N}$ there exists exactly one such semigroup with presentation $\langle u \mid u^n = u^{n+1} \rangle$. Together with the result from Theorem 6 this leads to the following theorem.

7 Theorem. *Let K be an arbitrary field. The graph $\mathcal{G}_{0,K}$ consists of a unique maximal coclass tree with root KZ_1. This tree is strongly virtually periodic with strong defect 1, strong period 1, and the single associated polynomial $f_{KZ_1}(x) = 1$.*

6. The graph $\mathcal{G}_{1,K}$

We determine the graph $\mathcal{G}_{1,K}$ for arbitrary fields K using the classification [2,3] of nilpotent semigroups of coclass 1. As a preliminary step, note that a nilpotent semigroup of coclass 1 has at least 3 elements. Up to isomorphism there exist exactly one semigroup of coclass 1 with 3 elements, namely Z_3, and nine semigroups with 4 elements.

8 Theorem. *Let K be an arbitrary field.*

(1) *The graph $\mathcal{G}_{1,K}$ consists of a unique maximal coclass tree $\mathcal{T}$ with root KZ_3 and corresponding infinite dimensional algebra $M_{K,1}$ (defined in (1)).*
(2) *The tree $\mathcal{T}$ is strongly virtually periodic with strong defect 2 and strong period 2. Let $A_1 \to A_2 \to \cdots$ denote the maximal infinite path of $\mathcal{T}$. For each algebra $B \in \mathcal{T}(A_2) \setminus \mathcal{T}(A_4)$ the polynomial corresponding to B has degree at most 1.*
   (a) *If $\sqrt{-1} \in K$, then $\mathcal{T}(A_2) \setminus \mathcal{T}(A_4)$ consists of A_2, 4 algebras with A_2 as parent, and 3 algebras with A_3 as parent; see the right box of Fig. 1.*
   (b) *If $\sqrt{-1} \notin K$, then $\mathcal{T}(A_2) \setminus \mathcal{T}(A_4)$ consists of A_2, 4 algebras with A_2 as parent, and 4 algebras with A_3 as parent; see the left box of Fig. 1.*

Proof. The first part of the statement is true by Theorem 6. To prove the second part we use the classification from [3]: there are the following $n + 2 + \lfloor n/2 \rfloor$ isomorphism types of semigroups of order n and coclass 1 for $n \geqslant 5$:

- $H_k = \langle u, v \mid u^{n-1} = u^n, uv = u^k, vu = u^k, v^2 = u^{2k-2} \rangle$, $2 \leqslant k \leqslant n-1$;
- $J_k = \langle u, v \mid u^{n-1} = u^n, uv = u^k, vu = u^k, v^2 = u^{n-2} \rangle$, $n/2 < k \leqslant n-1$;
- $X = \langle u, v \mid u^{n-1} = u^n, uv = u^{n/2}, vu = u^{n/2}, v^2 = u^{n-1} \rangle$, if $n \equiv 0 \bmod 2$;
- $N_1 = \langle u, v \mid u^{n-1} = u^n, uv = u^{n-1}, vu = u^{n-2}, v^2 = u^{n-2} \rangle$;
- $N_2 = \langle u, v \mid u^{n-1} = u^n, uv = u^{n-2}, vu = u^{n-1}, v^2 = u^{n-2} \rangle$;
- $N_3 = \langle u, v \mid u^{n-1} = u^n, uv = u^{n-1}, vu = u^{n-2}, v^2 = u^{n-1} \rangle$;
- $N_4 = \langle u, v \mid u^{n-1} = u^n, uv = u^{n-2}, vu = u^{n-1}, v^2 = u^{n-1} \rangle$.

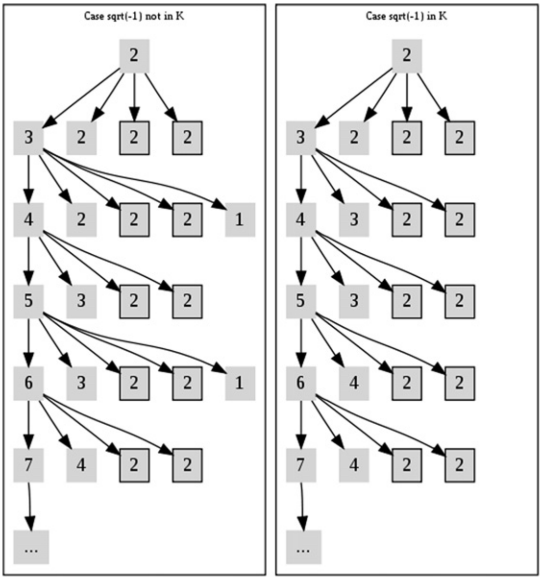

Fig. 1. Description of $\mathcal{T}(A_2)$ in $\mathcal{G}_{1,K}$ with root of dimension 3. Vertices with box correspond to non-commutative algebras. The polynomials of degree 1 are $2x+2$ and $2x+3$ for the two families on the infinite path and $x+2$ for the other two families.

We now show which of these semigroups give rise to isomorphic algebras.

- Show that $KH_2 \cong KH_k$ for $3 \leqslant k \leqslant n-1$ holds.
  Define $\mu : KH_2 \to KH_k$ via $\mu(u) = u + u^{k-1}$ and $\mu(v) = u + v$. As $(u + u^{k-1})^m = \sum_{i=0}^{m} \binom{m}{i} u^{m-i}(u^{k-1})^i = \sum_{i=0}^{m} \binom{m}{i} u^{m+(k-2)i}$ for $1 \leqslant m \leqslant n-1$, it follows that the elements $u+u^{k-1}$ and $u+v$ generate KH_k. The images of u and v under μ satisfy the relations of H_2 and hence μ induces an epimorphism. As KH_k and KH_2 have the same dimension, μ is an isomorphism.
- Show that $KJ_{n-1} \cong KJ_k$ for $n/2 < k < n-1$ and $KX \cong KJ_{n-1}$ if n is even and $\sqrt{-1} \in K$ holds.
  For the first part, define $\mu : KJ_{n-1} \to KJ_k$ via $\mu(u) = u$ and $\mu(v) = v - u^{k-1}$. For the second part, define $\mu : KX \to KJ_{n-1}$ via $\mu(u) = u$ and $\mu(v) = u^{n/2-1} - \sqrt{-1}v$. Then as above, μ is an isomorphism.
- Show that $KN_1 \cong KN_2$ and $KN_3 \cong KN_4$ hold.
  Note that (N_1, N_2) and (N_3, N_4) are pairs of anti-isomorphic semigroups. For each $i \in \{1, \dots, 4\}$, the subsemigroup $\langle u, u^{n-3} - v\rangle$ yields a basis of KN_i and is isomorphic to the dual semigroup of N_i. Hence $KN_1 \cong KN_2$ and $KN_3 \cong KN_4$ follow.

It remains to show that we have determined all isomorphisms. First, we consider KH_{n-1} and KJ_{n-1}. These are both commutative algebras; the first has an annihilator of dimension 2 generated by v and u^{n-2} and the second has an annihilator of dimension 1 generated by u^{n-2}. Hence the algebras are non-isomorphic. Secondly, we consider KN_1 and KN_3. These are both non-commutative algebras and they both have an annihilator of dimension 1; the first has a right annihilator of dimension 2 generated by v and u^{n-2} and the second has a right annihilator of dimension 1 generated by u^{n-2}. Hence the algebras are non-isomorphic. This proves our claim in the case $\sqrt{-1} \in K$ or n odd. In the case $\sqrt{-1} \notin K$ and n even, there is the additional algebra KX. This is a commutative algebra whose annihilator has dimension 1; hence we have to distinguish KX from KJ_{n-1}. Assume that $\mu : KX \to KJ_{n-1}$ is an isomorphism and denote $\mu(v) = av + \sum_{i=1}^{n-2} b_i u^i \in KJ_{n-1}$. Then $\mu(v)^2 = a^2v^2 + (\sum_{i=1}^{n-2} b_i u^i)^2$ in KJ_{n-1}, as $uv = vu = 0$ holds. Note that $v^2 = u^{n-2}$ in KJ_{n-1} and $\mu(v)^2 = \mu(v^2) = 0 \in KJ_{n-1}$ as $v^2 = 0$ in KX. An inspection of the coefficients now shows that $b_i = 0$ for $1 \leqslant i \leqslant n/2-2$.

The coefficient of u^{n-2} in $\mu(v)^2$ thus is $a^2 + b_{n/2-1}^2$. Since $\sqrt{-1} \notin K$, it follows that $a = b_{n/2-1} = 0$. This yields that $\mu(v) \in \langle u^{n/2}, u^{n/2+1}, \dots, u_{n-2}\rangle \leqslant (KJ_{n-1})^2$. Hence μ is not surjective, a contradiction.

We determine the edges of $\mathcal{G}_{1,K}$. Consider a semigroup S of order n from the above classification. In the quotient S/S^{n-2} the two elements u^{n-2} and u^{n-1} are identified, and hence the quotient is isomorphic to a semigroup of type H_k of order $n-1$. Note that the latter is valid for $n = 5$ also, as the semigroups H_k can be defined for order 4 as well.

The labels of the vertices in $\mathcal{G}_{1,K}$ follow immediately from the classification. This implies that $\mathcal{G}_{1,K}$ has strong defect 2 and strong period 2. (In fact, both values are minimal.) □

Images of the parts of $\mathcal{G}_{1,K,2}$ corresponding to semigroups of order at most 12 for $K = GF(p)$ with $p \leqslant 23$ together with the algorithms used to determine these graphs can be found at [4].

7. Computational experiments for $\mathcal{G}_{2,K}$

A classification of semigroups of coclass 2 is available in [2,3]. We used it to investigate $\mathcal{G}_{2,K}$ computationally, applying the isomorphism test for associative nilpotent algebras over finite fields in [6]. Semigroups of coclass 2 have a minimal generating set of size 2 or 3. We know from Section 4 that these two cases lead to independent subgraphs $\mathcal{G}_{2,K,2}$ and $\mathcal{G}_{2,K,3}$ of $\mathcal{G}_{2,K}$ which shall be considered separately.

We have determined the part of $\mathcal{G}_{2,K,2}$ corresponding to semigroups of order at most 12 for $K = GF(p)$ with $p \leqslant 23$, see [4]. We conjecture that for every field K the graph $\mathcal{G}_{2,K,2}$ has five maximal infinite paths which are described by the following infinite dimensional algebras:

- $\langle a, b \mid b^2 = ba = a^2b = 0\rangle$ with annihilator $\langle ab\rangle$;
- $\langle a, b \mid b^2 = ab = ba^2 = 0\rangle$ with annihilator $\langle ba\rangle$;
- $\langle a, b \mid b^3 = ab = ba = 0\rangle$ with annihilator $\langle b^2\rangle$;
- $\langle a, b \mid b^2 = aba = 0, ab = ba\rangle$ with annihilator $\langle ba\rangle$;
- $\langle a, b \mid b^2 = ba, ab = b^2a = 0\rangle$ with annihilator $\langle ba\rangle$.

Using these presentations to define semigroups with zero we obtain infinite semigroups that are annihilator extensions of the semigroup underlying $M_{K,1}$ and whose contracted semigroup algebras are the algebras defined by the presentations. If the conjecture on the number of infinite paths holds, then $\mathcal{G}_{2,K,2}$ contains five maximal coclass trees. Fig. 2 exhibits the respective trees of the computed graph for $K = GF(5)$. Our computational evidence suggests the following:

- the graph $\mathcal{G}_{2,GF(p),2}$ depends on p mod 4 only;
- the vertices in $\mathcal{G}_{2,GF(p),2}$ outside a maximal coclass tree have dimension 4 or 5;
- the roots of the maximal coclass trees of $\mathcal{G}_{2,GF(p),2}$ have dimension 4;
- each maximal coclass tree in $\mathcal{G}_{2,GF(p),2}$ is strongly virtually periodic; one tree has strong defect 1 and strong period 1, the other four trees have strong defect 2 and strong period 2;
- the strong defects and strong periods are independent of the field.

The polynomials describing the labels in the periodic parts of the maximal coclass trees have degree at most 1, a fact that follows from the classification in [2,3].

The graph $\mathcal{G}_{2,K,3}$ is known to consist of a single maximal coclass tree with root KZ_4 and infinite paths corresponding to $M_{K,2}$ by Theorem 6. We have determined the part of $\mathcal{G}_{2,K,3}$ corresponding to semigroups of order at most 12 for $K = GF(p)$ with $p \leqslant 5$, see [4]. In all three cases, the graph appears strongly virtually periodic with strong defect 2, and strong period 2. In accordance with the results from [2,3] the labels can be described by quadratic polynomials.

8. Computational experiments for $\mathcal{G}_{3,K}$

For the semigroups of coclass 3 there is no general classification known. We computed the semigroups of coclass 3 and order at most 17 up to isomorphism using the code provided in

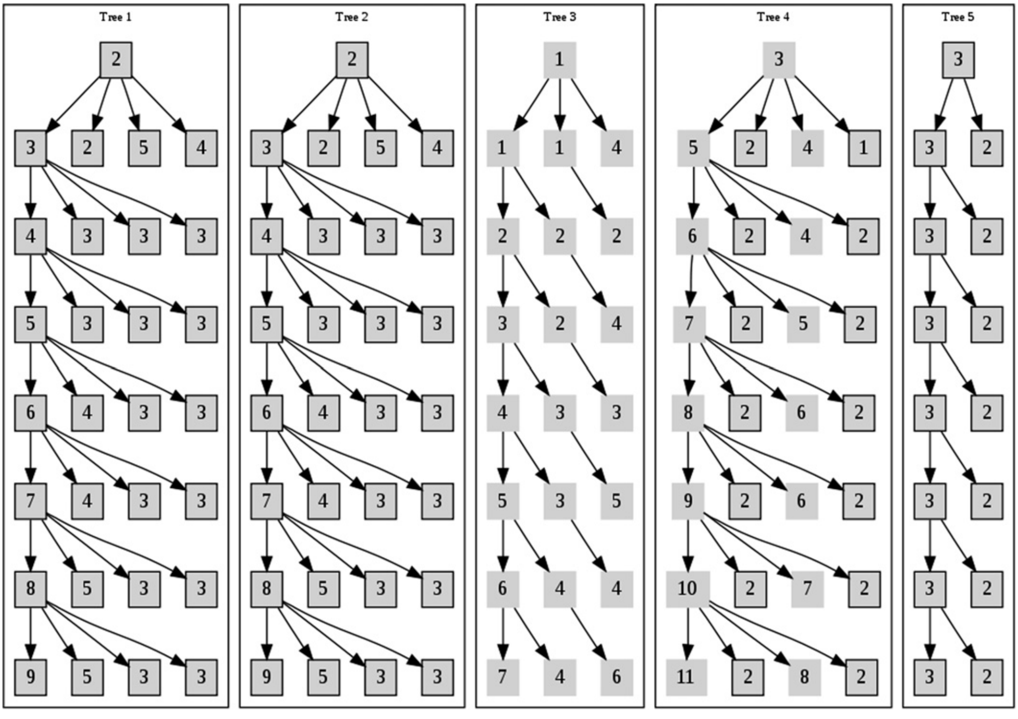

Fig. 2. Maximal coclass trees in $\mathcal{G}_{2,GF(5),2}$ up to depth 12.

[2, Appendix C]. Then we determined the part of $\mathcal{G}_{3,K,2}$ corresponding to these semigroups for $K = GF(p)$ with $p \leqslant 23$, see [4]. We summarise our observations:

- the graph $\mathcal{G}_{3,GF(p),2}$ depends on p mod 4 only;
- the graph $\mathcal{G}_{3,GF(p),2}$ has 15 maximal infinite paths of which 4 correspond to commutative algebras;
- the vertices in $\mathcal{G}_{3,GF(p),2}$ outside a maximal coclass tree have dimension 6, 7 or 8;
- the roots of the maximal coclass trees of $\mathcal{G}_{3,GF(p),2}$ have dimension 5, 6 or 7;
- the maximal coclass trees in $\mathcal{G}_{3,GF(p),2}$ are strongly virtually periodic with strong defect at most 3 and strong period at most 6;
- the strong periods are independent of the field;
- the polynomials describing the labels have degree at most 1.

These observations for $\mathcal{G}_{3,GF(p),2}$ are of particular interest as this is the first case in which some of the semigroups contain products of three elements that lie in different monogenic subsemigroups. In fact, $\mathcal{G}_{3,GF(p),2}$ has more complex features than all other graphs considered.

We have investigated the part of $\mathcal{G}_{3,K,3}$ corresponding to semigroups of order at most 12 for $K = GF(2)$ only. There appear to be 21 maximal infinite paths in $\mathcal{G}_{3,GF(2),3}$ with 5 paths corresponding to commutative algebras.

The graph $\mathcal{G}_{3,K,4}$ has 1 maximal infinite path corresponding to the commutative algebra $M_{K,3}$ by Theorem 6.

9. Concluding comments

Similar to the graphs $\mathcal{G}_{r,K}$, one can define a graph $\mathcal{G}_r$ whose vertices correspond one-to-one to the isomorphism types of semigroups of coclass r. Two vertices are joined by a directed edge $T \to S$

if $S/S^c \cong T$ where c is the class of S. It follows directly from [3, Lemma 3.1] that $\mathcal{G}_r$ does not have finite width (unless $r = 0$).

The additional use of the contracted semigroup algebras in the definition of $\mathcal{G}_{r,K}$ induces a dependence on the underlying field K, but it has the significant advantage that the graphs $\mathcal{G}_{r,K}$ seem to have finite width and exhibit periodic patterns which can be described in a compact way. Further, the field K seems to have no influence on the important aspects of the periodicity.

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